# Tutorial 5 (Feb 19, 21)

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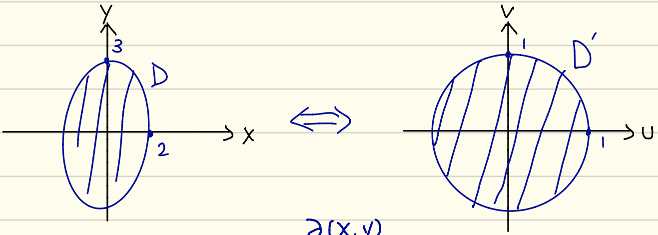
Q1) Evaluate $\iint_D x^2\,dA$, where $D$ is the region bounded by the ellipse $9x^2+4y^2=36$.

Sol) Step 1: Apply a change of variable $\begin{cases} x = 2u \\ y = 3v \end{cases}$.

then $9x^2+4y^2 \le 36 \iff 36u^2+36v^2 \le 36 \iff u^2+v^2 \le 1$

Picture:




Step 2: Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6.$$

Step 3: Apply the change of variables formula.

$$\iint_D x^2\,dx\,dy = \iint_{D'} 4u^2 \cdot (|6|\,du\,dv) = 24 \iint_{D'} u^2\,du\,dv$$

$$= 24 \int_0^{2\pi}\int_0^1 (r\cos\theta)^2 (r\,dr\,d\theta) = 24\left(\int_0^{2\pi} \cos^2\theta\,d\theta\right)\left(\int_0^1 r^3\,dr\right)$$

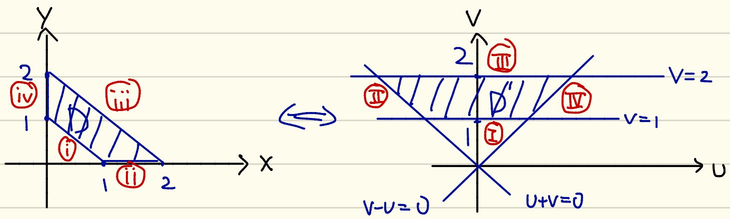
$$= 24\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 = 24\cdot\pi\cdot\frac{1}{4} = 6\pi$$

Remark: Alternatively, one may try another change of variables $\begin{cases} x = 2\rho\cos\phi \\ y = 3\rho\sin\phi \end{cases}$ where $0 \le \rho \le 1$, $0 \le \phi \le 2\pi$.

Q2) Evaluate $\iint_D \cos\frac{y-x}{y+x}\,dA$, where $D$ is the trapezoidal region with vertices $(1,0), (2,0), (0,2), (0,1)$.

Sol) Step 1: Apply a change of variables $\begin{cases} y-x = u \\ y+x = v \end{cases} \Longleftrightarrow \begin{cases} x = \frac{v-u}{2} \\ y = \frac{v+u}{2} \end{cases}$

Picture:




Boundary equations: $\begin{cases} \text{(i)} & x+y=1 \\ \text{(ii)} & y=0 \\ \text{(iii)} & x+y=2 \\ \text{(iv)} & x=0 \end{cases} \Longleftrightarrow \begin{cases} \text{(I)} & v=1 \\ \text{(II)} & u+v=0 \\ \text{(III)} & v=2 \\ \text{(IV)} & v-u=0 \end{cases}$

$\therefore D' = \{(u,v) \in \mathbb{R}^2 \mid 1 \le v \le 2,\ -v \le u \le v\}$

Step 2: Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Step 3: Apply the change of variables formula.

$$\iint_D \cos\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_{D'} \cos\frac{u}{v} \cdot \left(\left|-\frac{1}{2}\right| du\,dv\right) = \frac{1}{2}\int_1^2 \int_{-v}^{v} \cos\frac{u}{v}\, du\,dv$$

$$= \frac{1}{2}\int_1^2 \left[v \sin\frac{u}{v}\right]_{-v}^{v} dv = \frac{1}{2}\int_1^2 (v\sin 1 - (-v\sin 1))\,dv = (\sin 1)\cdot \int_1^2 v\,dv$$

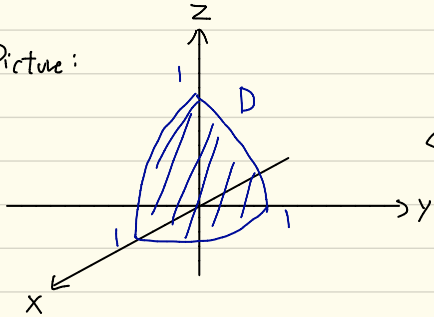
$$= \sin 1 \cdot \left[\frac{v^2}{2}\right]_1^2 = \frac{3}{2}\sin 1$$

Q3) Evaluate $\iiint_D x e^{x^2+y^2+z^2}\, dV$, where D is the portion of unit ball $x^2+y^2+z^2 \leq 1$ that lies in the first octant.

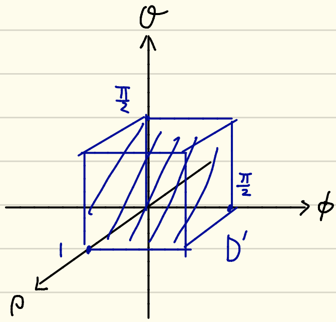
Sol) Adopting the method of change of variables using spherical coordinates:

Step 1: Apply a change of variables $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$

Picture:




$\Leftrightarrow$




$\therefore D' = \{(\rho,\phi,\theta) \in [0,+\infty)\times[0,\pi]\times[0,2\pi) \mid 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \frac{\pi}{2},\ 0 \leq \theta \leq \frac{\pi}{2}\}$

Step 2: Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$.

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2\sin\phi\left((-\sin\theta)(-\sin^2\phi\sin\theta - \cos^2\phi\sin\theta) - \cos\theta(-\sin^2\phi\cos\theta - \cos^2\phi\cos\theta)\right)$$

$$= \rho^2\sin\phi(\sin^2\theta + \cos^2\theta) = \rho^2\sin\phi$$

Step 3: Apply the change of variable formula.

$$\iiint_D x e^{x^2+y^2+z^2}\,dx\,dy\,dz = \iiint_{D'} (\rho\sin\phi\cos\theta)\, e^{\rho^2}\left(|\rho^2\sin\phi|\,d\rho\,d\phi\,d\theta\right)$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3 e^{\rho^2}\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi\right)\left(\int_0^1 \rho^3 e^{\rho^2}\,d\rho\right)$$

$$= [\sin\theta]_0^{\frac{\pi}{2}}\cdot\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\frac{\pi}{2}}\cdot\left(\left[\frac{\rho^2 e^{\rho^2}}{2}\right]_0^1 - \int_0^1 e^{\rho^2}\cdot d(\rho^2)\right)$$

$$= 1\cdot\frac{\pi}{4}\cdot\frac{1}{2}\left(e - [e^{\rho^2}]_0^1\right) = \frac{\pi}{8}$$

Remark: Evaluating the triple integral using spherical coordinates is the same as step 3.